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The Group which Contain less than Fifteen Operators of Order Two.

By G. A. Miller.

The groups which contain exactly thirteen operators of order two have recently been considered.[*] These are of somewhat special interest since they do not include any group whose order is a power of 2. For every smaller possible number (n) of operators of order 2 there is at least one group whose order is of the form 2^m. It is well known that every group (G) which contains at least one operator of order 2 contains an odd number of such operators. Hence we may assume that n represents successively all the odd numbers from 1 to 11.

When the operators of order two in G do not generate the entire group, they generate a characteristic subgroup H. The object of the present paper is a complete determination of H for every value of n within the given limits, and a determination of some properties of G which result directly from H. The determination of all the possible G's for any value of n is a much more difficult problem, which involves the determination of all groups of odd order. It is believed that a knowledge of all the possible H's for small values of n will be useful in the study of many other problems.

When $n = 1$, H is the cyclic group of order 2. When $n = 3$, two of these operators generate a dihedral rotation group which includes the third. Hence, H is either the symmetric group of order 6 or the four-group, as the operators of order two are non-commutative or commutative. Each of these two groups has the symmetric group of order 6 for its group of isomorphisms. They are so well known that it seems unnecessary to specify other properties.

Since $5 \equiv 1 \pmod 4$ there are exactly two groups whose order is of the form 2^m and which contain only 5 operators of order 2;[†] viz., the octic group and the group of order 16 which involves operators of order 8 that are transformed into their third powers. The former of these is generated by its operators of order 2, while these operators generate the octic subgroup of the latter. Hence there is only one group of order 2^m which contains only five operators of order 2 and is generated by these operators, viz., the octic group, or the group of the square. It remains to determine the possible H whose order is not of the form 2^m.

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[*] Bulletin of the American Mathematical Society, Vol. XII, 1906, p. 289.

[†] Transactions of the American Mathematical Society, Vol. 6, 1905, p. 62.

Two of the 5 operators of order 2 generate a dihedral rotation group whose order cannot exceed 10. If it is 10 any pair of operators of order 2 generate it. If it were 6 the two remaining operators of order 2 could not transform this subgroup into itself since a group of order 12 cannot involve exactly 5 operators of order 2. Nor could one of these operators transform this subgroup into a different subgroup since the latter would have at least two operators of order 2 which would not be in the former subgroup. The product of each pair of operators of order 2 could not be of order 2 since $5 \equiv 1 \bmod 2^a$. Hence, *the octic group and the dihedral rotation group of order* 10 *are the only two groups which contain exactly* 5 *operators of order two and are generated by these operators.*

§ 1. *General Theorems on Dihedral Rotation Groups.*

The following general theorems are of interest in themselves and will greatly simplify the consideration of the remaining cases.

THEOREM I. *If an operator* (t) *of order two transforms into itself a dihedral rotation group* (D) *whose order* (d) *is not divisible by* 4, *the group* $[D, t]$ *generated by* D *and* t *is the direct product of two dihedral rotation groups.**

To prove this theorem we shall first consider the special case when $d = 2p^a$, where p is an odd prime number. Let C represent the cyclic subgroup of order p^a contained in D. Since the group of isomorphisms of C is cyclic it follows that t transforms C according to an operator in the group of cogredient isomorphisms of D. That is, t transforms either each operator of C into its inverse or it is commutative with each of these operators. In the latter case it is also commutative with the operators of order 2 in D since it could not transform one of these into itself multiplied by an operator of odd order with which it is commutative. In the former case it is commutative with an operator of order 2 in D since it cannot have d conjugates under D. Hence t is always commutative with at least one operator of order 2 in D. As the product of these two operators is again of order 2 and could be used for t, it follows that $[D, t]$ *is the direct product of* D *and a group of order* 2 *whenever* $d = 2p^a$. This direct product is the dihedral rotation group of order $4p^a$.

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*One of these dihedral rotation groups may be of order 2. In this case $[D, t]$ is the direct product of D and a group of order 2.

When t is commutative with any operator, besides the identity, in a Sylow subgroup of C it is commutative with all the operators of this Sylow subgroup. If t is non-commutative with any such operator it transforms all the operators of this Sylow subgroup into their inverses. In the latter case, t is commutative with an operator of order 2 in a dihedral rotation group containing this Sylow subgroup and being contained in D, since the number of such dihedral rotation groups is odd. In the former case, t is commutative with all the operators of such a dihedral rotation group for the same reason. Hence we have as before that t is commutative with at least one operator of order 2 contained in D.

The Sylow subgroups of C may be divided into two sets; the first being composed of those whose operators are commutative with t while the second includes those whose generators are transformed into their inverses by t. The direct product of the two groups generated by these two sets is C, and D may be constructed by dimidiating the two dihedral rotation groups which involve respectively these two factor groups of C. The number of operators of order 2 in D with which t is commutative is therefore equal to the order of the former of the two given factor groups of C, and *the number of operators of order* 2 *in* $[D, t]$ *which are not also in* D *is equal to the sum of the orders of these two factor groups*. Moreover, it is clear that t may be replaced by some operator of order 2 in the second of the two groups whose dimidiation gives rise to D. In other words, $[D, t]$ is the direct product of these two groups. This proves the theorem in question.

For our present purpose the possible number of operators of order 2 in $[D, t]$ is most important. From what precedes it follows that this number is $d/2$ plus the sum of any two relatively prime numbers whose product is $d/2$. In particular, if one of these two numbers is $d/2$ the other is unity. This special case leads to the maximum number of operators of order 2 in $[D, t]$ and is the only possible case when $d/2$ is a power of a prime, as was observed above. The main result may be expressed as follows: *the number of operators of order* 2 *in* $[D, t]$ *is* $d/2 + f_1 + f_2$, *where* f_1, f_2 *are relatively prime and* $f_1 f_2 = d/2$. The operator t may be so chosen that f_1, f_2 have any values satisfying the given conditions. If $f_1 = d/2$ it must be assumed that $f_2 = 1$.

THEOREM II. *If the order* (d) *of a dihedral rotation group* (D) *is divisible by* 4, *there is an operator* (t) *of order* 2 *such that* $t^{-1} D t = D$ *and* $[D, t]$ *contains exactly* $d/2 + 3$ *operators of order* 2.

To obtain this group t must be chosen so as to be commutative with each operator of the cyclic subgroup of order $d/2$ contained in d and to transform the remaining operators into themselves multiplied by the operator of order 2 in this cyclic subgroup. The operators of $[D, t]$ which are not in D nor in the direct product of the cyclic subgroup of order $d/2$ contained in D and t are all of order 4 and have a common square. This is the only possible case in which a dihedral rotation group, extended by means of an operator of order 2, gives rise to a group which contains only two more operators of order 2 than the original group.

THEOREM III. *The group of isomorphisms of the dihedral rotation group of order* d *is the holomorph of the cyclic group of order* $d/2$, *except when* $d = 4$.*

When d is not divisible by 4 this theorem requires no proof since the group of cogredient isomorphisms of the holomorph of the cyclic group of order $d/2$ is the group of isomorphisms of the subgroup which is the dihedral rotation group (D) of order d. In general, the non-invariant operators of order 2 in D may be transformed according to the operators of the cyclic group of order $d/2$, without affecting the operators of the cyclic subgroup of this order. Hence the group of isomorphisms of D contains an invariant cyclic subgroup of order $d/2$. It can be represented as a substitution group on $d/2$ letters and includes operators which transform this cyclic subgroup in every possible manner. This completes the proof since D is invariant in the holomorph of the cyclic group of order $d/2$.

THEOREM IV. *If an operator* t *of order* 2 *transforms a dihedral rotation group* (D) *into itself, and is commutative with all the operators of a Sylow subgroup of order* 2^m *contained in* D, *then* $[D, t]$ *is the direct product of two dihedral rotation groups.*

Since t is commutative with a non-invariant operator t_1 of order 2 contained in D, it transforms into itself some dihedral rotation group involving any Sylow subgroup of the cyclic group of order $d/2$ contained in D. The product of these Sylow subgroups whose generators are transformed into their inverses by t together with t generates a dihedral rotation group which is invariant under $[D, t]$.

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* The group of Theorem II. corresponds to the invariant operator of order 2 in this holomorph. Jordan, Traité des substitutions, 1870, p. 60.

The product of the remainder of these Sylow subgroups and tt_1 generate a second dihedral rotation group. As all the operators of one of these groups are commutative with each operator of the other it is clear that $[D, t]$ is their direct product. It may be observed that Theorem I can be directly derived from this theorem. On account of its importance it seemed desirable to prove it separately.

Suppose that the order of D is 2^m. If t transforms into itself each operator of the cyclic group (C) of order 2^{m-1} contained in D then $[D, t]$ is either the direct product of D and t, or it is the group of Theorem II. In the former case it contains $2^m + 3$ operators of order 2, while there are $2^{m-1} + 3$ such operators in the latter case. If t transforms each operator of C into its inverse, $[D, t]$ is either the given direct product or the dihedral rotation group of order 2^{m+1}. The latter contains $2^m + 1$ operators of order 2. The only other case which requires consideration is when t transforms every operator of C into its $2^{m-2} + 1$ power, $m > 3$. There are two conjugate operators of order 2 in the group of isomorphisms of D which have this property. If t represents one of these, $[D, t]$ contains $2^{m-1} + 2^{m-2} + 3$ operators of order 2. Hence the following:

Theorem V. *There are exactly four groups of order* 2^{m+1} *which result by extending the dihedral rotation group of order* 2^m, $m > 3$ *by means of operators of order* 2. *These contain respectively* $2^{m-1} + 3$, $2^{m-1} + 2^{m-2} + 3$, $2^m + 1$, *and* $2^m + 3$ *operators of order* 2.

The second of these four groups does not exist as a distinct group when $m < 4$, when $m = 2$ the only possible groups are the octic and the abelian of type (1, 1, 1). If D represents any dihedral rotation group at least one of its Sylow subgroups is transformed into itself by t, whenever $t^{-1} D t = D$ and $t^2 = 1$. Hence the number of operators of order 2 in $[D, t]$ which are not also in D cannot be less than the difference between $2^{m-1} + 1$ and one of the four numbers given in Theorem V.

Suppose that G contains two dihedral rotation groups of order $2p$, p being a prime, and that these are conjugate under an operator (t) of order 2 contained in G. We proceed to prove that G contains at least $2p + 3$ operators of order 2. As this has been proved above for the case when $p = 2$ we shall assume that p exceeds 2 in what follows. To prove the theorem in question it is only necessary to prove that G cannot contain exactly $2p + 1$ operators of order 2. We proceed to prove this fact.

Let s_1, s_2 represent two operators of order 2 which generate one of the two conjugate dihedral rotation groups of G and let D_1 represent this group. Since G contains only $p + 1$ operators of order 2 besides those of D, it follows that $s_1 s_2$ transforms at least one of these (s_3) into itself. Hence s_3 transforms into itself the cyclic subgroup of order p contained in D_1. It cannot transform D_1 into itself as G contains an operator of order 2 which transforms D_1 into another group (D_2). Hence we may suppose that $s_3^{-1} D_1 s_3 = D_2$ and that D_1 and D_2 have p common operators, and therefore involve $2p$ distinct operators of order 2. As $s_3^{-1} s_1 s_3 = s_4$ is different from s_3 and is also commutative with the operators of order p contained in D_1, it has been proved that G cannot contain exactly $2p + 1$ operators of order 2. Hence there results

THEOREM VI. *If a group contains two dihedral rotation groups whose order is twice a prime number* (p) *and if these two groups are conjugate under one of its operators of order two, then the group contains at least* $2p + 3$ *operators of order* 2.

§ 2. *Groups which Contain exactly seven Operators of Order Two.*

When $n = 7$ the dihedral rotation group generated by two of these operators (s_1, s_2) has one of the following numbers for its order: 4, 6, 8, 10, 12, 14. In the last two cases the group generated by s_1, s_2 includes the other five operators of order 2. Hence it remains to consider the cases when the order of $s_1 s_2$ does not exceed 5. From Theorems I and VI it follows directly that the order of $s_1 s_2$ cannot be 5.

If $s_1 s_2$ is of order 4 the octic group is generated by s_1, s_2. One of the remaining two operators of order 2 transforms this into itself as two octic groups have to contain at least 7 distinct operators of order 2. Hence the seven operators of order 2 generate the group of order 16 which involves exactly three octic groups. If $s_1 s_2$ were of order 3 an operator of order 2 which is not in $[s_1, s_2]$ would transform this group into itself according to Theorem VI. Hence the seven operators of order 2 would generate the dihedral rotation group of order 12 mentioned above. Finally, if the product of every two of these seven operators is of order 2, they generate the abelian group of order 8 and of type (1, 1, 1). Combining these results we have the following theorem: *If a group contains exactly seven operators of order* 2 *and is generated by these operators its order is* 8, 12, 14 or 16. In each case there is only one group. The first is abelian, the second and third are dihedral rotation groups, while the fourth is the only non-abelian group of order 16 which contains just three pairs of conjugates of order two. Every group which contains exactly 7 operators of order 2 must therefore contain one of these four groups as a characteristic subgroup.

§ 3. *Groups which Contain exactly nine Operators of Order Two.*

Since $9 \equiv 1 \bmod 4$ the dihedral rotation group of order 16 is the only group of order 2^m which contains exactly 9 operators of order 2 and is generated by these operators, and there is only one other group whose order is of the form 2^m and which contains exactly 9 operators of order 2. Before proceeding to the consideration of the groups whose orders are not of the form 2^m it seems desirable to prove a theorem which will be frequently employed in what follows.

THEOREM. *If a Sylow subgroup of any group* (G) *contains more than one subgroup of prime order* (p) *but does not contain all the subgroups of this order contained in G, then there are at least p^2 subgroups of order p in G besides those in a given Sylow subgroup.*

Before giving a proof of this theorem we shall give one illustration. The symmetric group of order 24 contains a Sylow subgroup of order 8 which contains five operators of order 2, but does not include all of the operators of this order in the symmetric group. The given theorem says that the number of additional operators of this order is at least 4. This gives the exact number of operators of order 2 in the symmetric group of order 24, viz. 9.

To prove the theorem we let S_1, S_2 represent two Sylow subgroups of G such that they have the largest possible number of common operators of order p without having all the operators of this order in common. These common operators generate a common subgroup which is transformed into itself by some other operator of order p in S_1. This operator transforms S_2 into p conjugates such that each has at least p subgroups of order p which are not in any of the others not contained in S_1. This proves the theorem.

From this theorem it follows directly that a Sylow subgroup of the groups in question contains at most five operators of order 2. Let s_1, s_2 represent any two operators of order 2 contained in the required group. There is clearly one and only one group in which the order of $s_1 s_2$ is either 8 or 9. The former of these was considered above. The order of $s_1 s_2$ cannot be 5 or 7 according to theorems I and VI. If this order is 6 each of the remaining operators of order 2 transforms the dihedral rotation group of order 12 into itself and H is the group of theorem II. Since the order of $s_1 s_2$ cannot be 2 for every value of s_1, s_2 it remains only to consider the cases when this order is either 3 or 4.

If $s_1 s_2$ is of order 4 the octic group (O_1) generated by s_1, s_2 involves five of the operators of order 2. Let s_3 represent one of the remaining four operators of this order. The transform (O_2) of O_1 with respect to s_3 is distinct from O_1 and has three operators of order 2 in common with O_1. We may suppose that s_1 is common to O_1 and O_2 while s_2 does not have this property. The transform (O_3) of O_2 with respect to s_2 contains s_3 and the remaining operator of order 2. These three octic groups are generated by their common operators and s_2, $s_2^{-1} s_3 s_2$, s_3 respectively. Hence they are transformed by G according to the symmetric group of order 6.

Since only two operators of order 2 in G correspond to an operator of order 2 in this symmetric group, the order of $s_2 s_3$ is either 3 or 6. The latter was considered above so that we may assume this order to be 3. The four-group which is common to O_1, O_2, O_3 together with $s_2 s_3$ generate the alternating group of order 12. As this is transformed into itself by s_2, the nine operators of order 2 generate the symmetric group of order 24. This symmetric group could therefore be defined as a group which contains exactly 9 operators of order 2 such that the order of the product of any two does not exceed 4 while there are at least two whose product is of this order.

If $s_1 s_2$ is of order 3 the group (D_1) generated by s_1, s_2 contains only three operators of order 2. Suppose that G contains another operator of order 2 s_3 which transforms D_1 into itself. As the dihedral rotation group $[D_1, s_3]$ contains operators of order 6 which are generated by two operators of order 2, the resulting group was considered above. Hence it may be assumed that no operator of order 2 contained in G transforms D_1 into itself. Let $s_3^{-1} D_1 s_3 = D_2$.

If D_1, D_2 have a common subgroup of order 3 they have no operator of order 2 in common and the 9 operators of order 2 are contained in three conjugate subgroups of order 6. As these are permuted by H according to the symmetric group of order 6, it follows that $s_1 s_3$ is an operator of order 3, which is commutative with the operators of order 3 in D_1. Hence H is the group of order 18 which is obtained by extending the non-cyclic group of order 9 by an operator transforming each of its operators into its inverse.

Suppose that G contains two operators of order 2 which constitute a complete set of conjugates. As these would generate an invariant four-group, G would involve an octic group and hence it would contain two operators of order 2 whose product would exceed 3. As these cases have been considered it follows that no two operators of order 2 form a complete set of conjugates under G in the cases which remain to be considered. Similarly, it is easy to see that three operators of order 2 could not form a complete set of conjugates in these cases since neither a four-group nor a symmetric group of order 6 can be invariant. If there were a complete set of four conjugates, they would be composed of two pairs of commutative operators and hence they would generate the octic group.

Five, six, or seven operators of order 2 could not form a complete set of conjugates without leading to groups already considered, since the remaining operators of order 2 could not all be invariant. If there were a complete set of eight conjugates of order 2, there would be an invariant operator of this order. Hence all the operators of order 2 would be contained in four conjugate four-groups. Each of these would transform one of the others into itself and hence G would contain the octic group. Hence the nine operators of order 2 in the cases which remain to be considered form one set of conjugates.

Hence H is isomorphic with a transitive substitution group (T) of degree 9 which has exactly 9 operators of order 2 and these also form a complete set of conjugates. The subgroup (K) which corresponds to the identity in T is of odd order and each of its operators is commutative with all the operators of H. The transitive groups of degree 9 are well known and it is easy to verify that only two of them are generated by such a set of 9 operators of order 2, viz., the two groups of order 18 which are of class 8. In verifying this statement it may be convenient to observe that T can contain no substitution of even order and of degree 9, and that the product of any two of its substitutions of order 2 is of order 3 and of degree 9.

The cases when H is simply isomorphic with T have been considered. We may therefore assume that K is not the identity. Suppose that s_3 is an operator of order 2 which is not in $[s_1, s_2]$. From T it follows that $s_3 s_1 s_2 s_3 = s_2 s_1 t$, where t is an operator of K. If $s_1 s_2$ and $s_1 s_3$ were commutative they would generate a group of order 9 and the 9 operators of order 2 would generate a group of order 18. Hence $s_1 s_2$, $s_1 s_3$ generate the non-commutative group of order 27 which involves 26 operators of order 3, since their commutator is an invariant operator of order 3. As s_1 is commutative with only three operators in this group of order 27, the nine operators of order 2 in question generate this group of order 54.

Hence, *there are exactly six groups which contain only* 9 *operators of order* 2 *and are generated by these operators.* The orders of these groups are: 16, 18, 18, 24, 24, 54. The first two are dihedral rotation groups, while the fourth is a symmetric group. The third is composed of the abelian non-cyclic group of order 9 and operators which transform each operator of this abelian group into its inverse. The fifth is the group of theorem II, § 1. The last is simply isomorphic with one of the transitive groups of degree 9. It contains 26 operators of order 3, 18 of order 6, and 9 of order 2 in addition to the identity.

§ 4. *Groups which Contain exactly eleven Operators of Order Two.*

Let s_1, s_2 represent two such operators. The order of $s_1 s_2$ cannot exceed 11. If it is either 10 or 11 the group is the dihedral rotation group of order 20 or 22. The order of $s_1 s_2$ could not be 9 according to theorem I § 1. If $s_1 s_2$ is of order 8, H is the group of theorem II § 1. This group is also completely defined by the facts that its order is 32 and that it contains exactly eleven operators of order 2, such that the order of the product of two of them is 8*. The order of $s_1 s_2$ cannot be 7 according to theorems I and VI.

If $s_1 s_2$ is of order 6, the dihedral rotation group generated by s_1, s_2 is invariant under H since a group of order 6 cannot involve more than three operators of order 2 and hence H cannot involve two such dihedral rotation groups which are conjugate under one of its operators of order 2. Let s_3 be any operator of order two contained in H but not in $[s_1, s_2]$. From the facts that $[s_1, s_2]$ contains a characteristic operator of order 2 and that H contains only 11 operators of this order it follows that $[s_1, s_2, s_3]$ is the group of theorem II § 1. Its order is 24 and it involves 9 operators of order 2. One of the remaining operators of order two (s_4) transforms this group of order 24 into itself and is commutative with 12 of its operators. As these can include only one operator of order 2 this isomorphism is again completely determined and $[s_1, s_2, s_3, s_4]$ is of order 48. This group is completely defined by its order and the fact that it involves just 11 operators of order 2 which generate it.

The case when s_1, s_2 is of order 5 need not be considered since it follows from theorem VI § 1 that H could not contain two dihedral rotation groups of order 10 which are conjugate under one of its operators of order 2. The dihedral rotation group of order 10 is therefore invariant and H is the direct product of this subgroup and a subgroup of order 2. In other words, H is the dihedral rotation group of order 20 since the direct product of the dihedral rotation group of order $2p$ and a group of order 2 is the dihedral rotation group of order $4p$.

It remains to consider the cases when the order of the product of any two operators of order two in G cannot exceed 4. If s_1s_2 is of order 4 the octic group $[s_1, s_2]$ may transform one of the remaining operators of order 2 into itself. In this case H is the only group of order 16 which contains exactly 11 operators of order 2. Since six operators of G are not in $[s_1, s_2]$ there is at least one pair of conjugates under this group. Let s_3 be one of such a pair. The subgroup of order 4 in $[s_1, s_2]$ which transforms s_3 into itself is cyclic, otherwise H would involve the group of order 8 and of type (1, 1, 1) as an invariant subgroup and hence would be the group of order 16 which has been considered.

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* Quarterly Journal of Mathematics, Vol. 28, 1896, p. 241, No. 21.

If s_3 does not transform $[s_1, s_2]$ into itself it transforms it into an octic group which involves the same cyclic subgroup of order 4. An operator of order 2 (s_4) in this octic group but not in $[s_1, s_2]$ must transform $[s_1, s_2]$ into itself since G contains only 11 operators of order 2. Hence $[s_1, s_2]$ is transformed into itself either by s_3 or by s_4. In the latter case, G would contain an operator of order 2 not in $[s_1, s_2]$ and commutative with the operators of order 4 in $[s_1, s_2]$ since the order of the product of two operators of order 2 cannot exceed 4. We may therefore assume that s_3 transforms $[s_1, s_2]$ into itself. Hence $[s_1, s_2, s_3]$ is the group of order 16 which contains just 7 operators of order 2 and is generated by them.

Let s_5 represent any operator of order 2 in G but not in $[s_1, s_2, s_3]$. If s_5 did not transform $[s_1, s_2, s_3]$ into itself it would transform it into a group having an octic subgroup in common. An operator of order 2 in this group but not in the octic common subgroup would transform $[s_1, s_2, s_3]$ either into itself or into another group having this common octic subgroup. The latter case is impossible since the six operators of order 2 not in the common octic subgroup would constitute three pairs of conjugate operators which H would transform according to the symmetric group of order 6. As H would contain an invariant operator of order 2, viz., the characteristic operator of this order in the common octic subgroup, this would require that the product of two operators of order 2 in H would have an order larger than 4. Hence it may be assumed that s_5 transforms $[s_1, s_2, s_3]$ into itself.

If s_5 is so chosen as to be commutative with each operator of the quaternion group contained in $[s_1, s_2, s_3]$ while it tranforms each of the other operators into themselves multiplied by the operator of order 2 in this quaternion group the resulting group will involve 4 operators of order 2 and 12 of order 4 besides those of $[s_1, s_2, s_3]$. Hence this group of order 32 is distinct from the one considered above and is also generated by 11 operators of order 2. As there are only two groups of order 32 which contain exactly 11 operators of order 2 and are generated by these operators it has been proved that H is this group of order 32 whenever $s_1 s_2$ is of order 4 and no two of its operators of order 2 give a product of larger order.

It remains to consider the cases when the order of any two operators of order 2 in G does not exceed 3. If $s_1 s_2$ is of order 3 there must be at least two operators s_3, s_4 of order 2 which are commutative with $s_1 s_2$. If there are only two such operators they are conjugate under s_1 since G cannot contain the direct product of $[s_1, s_2]$ and a group of order 2. Hence $s_3 s_4$ is of order 3, commutative with $s_1 s_2$, and transformed into its inverse by s_1. This is impossible since s_1 would transform into their inverses all the operators of $[s_1 s_2, s_3 s_4]$. If $s_1 s_2$ were commutative with exactly five operators of order 2 one of these would also be commutative with s_1 and hence H would involve the direct product of $[s_1, s_2]$ and a subgroup of order 2.

As this is again impossible, it remains only to consider the case when $s_1 s_2$ is commutative with eight of the operators of order 2. Two of these s_3, s_4 would be conjugate under s_1 and hence $s_3 s_4$ would be transformed into its inverse by s_1. If $s_3 s_4$ is of order 3 this is the case considered above. If it is of order 2 it is commutative with s_1 and hence has also been considered. From this we conclude that *if a group contains exactly* 11 *operators of order* 2 *it must contain at least two such operators whose product is of an order which exceeds* 3, and that there are exactly six groups which involve only 11 operators of order 2 and are generated by them.

SUMMARY.

Combining the results of the article "*The Groups which Contain thirteen Operators of Order Two*" (Bulletin of the Am. Mathematical Society, Vol. XII, 1906, p. 289) with those of the present article, we arrive at the following table, where n represents the number of operators of order 2 in the groups which are generated by these n operators.

n			*Orders*			
1	2					
3	4	6				
5	8	10				
7	8	12	14	16		
9	16	18	18	24	24	54
11	16	20	22	32	32	48
13	24	26	40	48	72	